Rediscovery of the S-matrix and factorization

Based on the lecture "Monte Carlo Event Generators" by Klaus WERNER, given at the summer school "Heavy Ion Collisions in the QCD phase diagram" , June 27 - July 08, 2022, Nantes, France.

Factorization

The most popular approach to treat HE pp, is based on "factorization", where the di-jet cross section is given as

$$\sigma_{\mathrm{dijet}} = \sum_{kl} \int \frac{d^3p_3 d^3p_4}{E_3 E_4} \int dx_1 dx_2\, f^k_{\mathrm{PDF}}(x_1,\mu_F^2)\, f^l_{\mathrm{PDF}}(x_2,\mu_F^2)$$

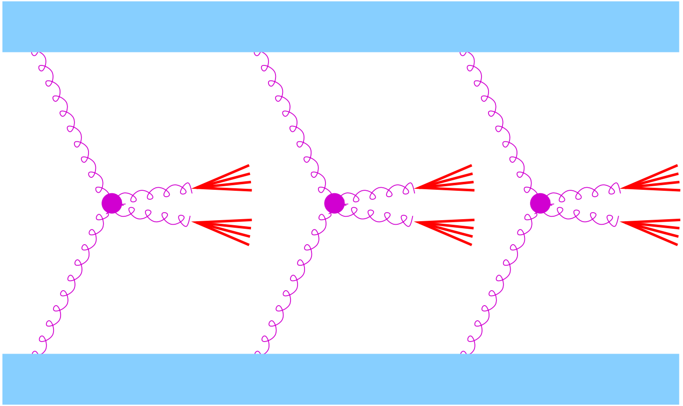
$$\frac{1}{32 s \pi^2} \bar{\sum} |\mathcal{M}|^2 \delta^4(p_1 + p_2 - p_3 - p_4),$$

Easy! No sophisticated MC needed.

But where are these complicated "parallel" scatterings?

The di-jet cross section is an inclusive cross section, i.e. one counts di-jets, not di-jet events, so a *N*-di-jet event counts *N* times

Here: N=3: We count 3 dijets.



Summing N-di-jet events, we get for the inclusive di-jet cross section

$$\sigma_{\mathbf{dijet}} = \sum_N N\, \sigma_{\mathbf{dijet}}^{(N)}$$

whereas the total cross section (forgetting soft for the moment)

$$\sigma_{\mathbf{tot}} = \sum_N \sigma_{\mathbf{dijet}}^{(N)}$$

For inclusive cross section, enormous simplifications apply!

To understand this we have to first look closer at "parallel scattering", using an appropriate tool (S-matrix approach).

Crash course on S-matrix theory

S-matrix theory is based on two major beliefs:

- **Even when a theory is not 100% known, one may obtain considerable guidance from a general quantum mechanical framework based on (plausible) hypotheses**

- **Properties of functions *f(x)* of real variables are much better understood when "continuing" into the complex plane** *)

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*) based on the uniqueness of analytic continuation in the complex plane, an extremely power theorem

Reminder: The scattering operator $\hat{S}$ is defined via

$$|\psi(t=+\infty)\rangle = \hat{S}\,|\psi(t=-\infty)\rangle$$

The S-matrix is the corresponding representation

$$S_{ij} = \langle i|\,\hat{S}\,|j\rangle$$

for basis states $|i\rangle$ and $|j\rangle$.

The T-matrix is defined as

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta(p_f - p_i)T_{fi}$$

Fundamental properties of the S-matrix

Most important:

The scattering operator $\hat{S}$ must be unitary:

$$\hat{S}^{\dagger}\hat{S} = 1$$

(elementary quantum mechanics), which means the scattering does not change the normalisation of a state.

Very plausible 3 hypotheses:

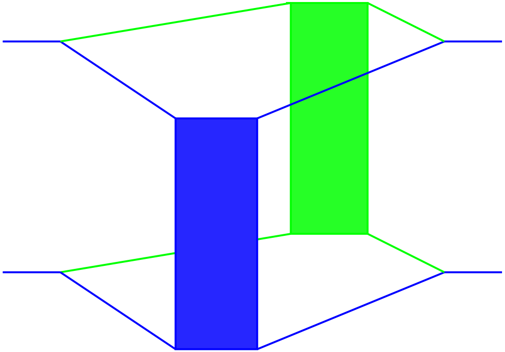
- ☐ T_{ii} **is Lorentz invariant** $\rightarrow$ **use** s, t
- ☐ $T_{ii}(s,t)$ **is an analytic function of** s, **with** s **considered as a complex variable** **(Hermitean analyticity)**
- ☐ $T_{ii}(s,t)$ **is real on some part of the real axis**

Using the Schwarz reflection principle (a theorem), $T_{ii}(s,t)$ **first defined for** Im$s \geq 0$ **can be continued in a unique fashion via** $T_{ii}(s^*,t) = T_{ii}(s,t)^*$.

In the following we use $T = T_{ii}$ **(elastic scattering).**

Based on the observation of multiple scattering at HE, and the necessity of parallel scatterings, one postulates a particular form of the (elastic scattering) **T-matrix** *):

$$-i\,\{iT_{\mathrm{Pom}} \times ... \times iT_{\mathrm{Pom}}\}$$



The elements are called Pomerons (just a name)

Compatible with pQCD, hidden in the "boxes" (Pomerons)

*) simplified version, without energy conservation

It can be shown (from unitarity + 3 hypotheses):

$$2s\,\sigma_{\mathrm{tot}} = (2\pi)^4\delta(p_f - p_i)\sum_f |T_{fi}|^2 = \frac{1}{\mathrm{i}}\mathrm{disc}\,T$$

Interpretation: $\frac{1}{\mathrm{i}}\mathrm{disc}\,T$ can be seen as a so-called "cut diagram", with modified Feynman rules, the "intermediate particles" are on mass shell.

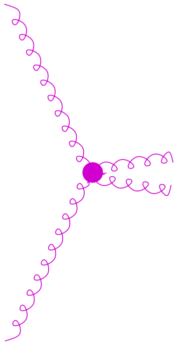
Cut diagrams ($\frac{1}{\mathrm{i}}\mathrm{disc}\,T$) represent inelastic processes, uncut diagrams (T) elastic ones.

The notion of "cutting" is extremely useful in our approach, more details later.

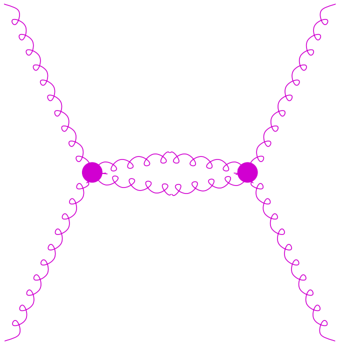
Example: Di-jet diagram

(We consider here jet = parton)

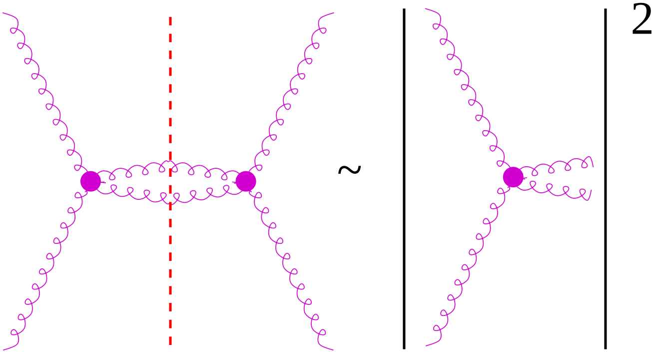
Inelastic diagram:



Elastic diagram:



The cut diagram is (up to constant) equal to squared inelastic one



The cut diagram ($\frac{1}{i}$disc T) represents inelastic processes, for T representing the corresponding elastic ones.

Why does factorization work ?

Easy to see in our **S-matrix approach** based on parallel scatterings (Gribov-Regge picture, as used in EPOS).

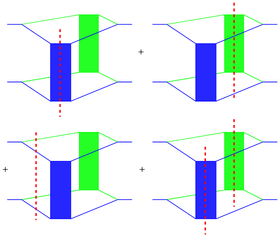
Here, the simplified version, without energy conservation, using simple assumptions:

Consider multiple scattering amplitude, i.e. a T-matrix of the form

$$iT = \prod iT_{\mathrm{P}}$$

T_P represents one elementary scattering (Pomeron)

Cross section:
sum over all
cuts.



Here, two parallel
scatterings

A cut Pomeron is (up to a constant) equal to the inelastic amplitude squared, it represents the weight to produce a di-jet.

An uncut Pomeron is just an elastic scattering, nothing produced.

For each cut Pom:

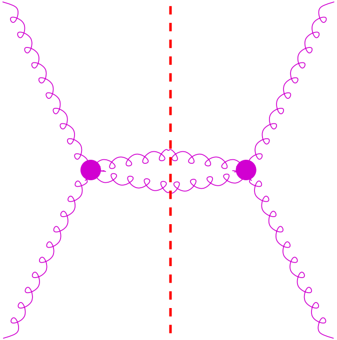
$$\frac{1}{i}\mathrm{disc}T_{\mathrm{P}} = 2\mathrm{Im}T_{\mathrm{P}} \equiv G$$

For each uncut one (considering imaginary T_{P}):

$$\begin{aligned} & iT_{\mathrm{P}} + \{iT_{\mathrm{P}}\}^{*} \\ = \; & i\,(i\,\mathrm{Im}T_{\mathrm{P}}) + \{i\,(i\,\mathrm{Im}T_{\mathrm{P}})\}^{*} \\ = \; & -2\mathrm{Im}T_{\mathrm{P}} \equiv -G \end{aligned}$$

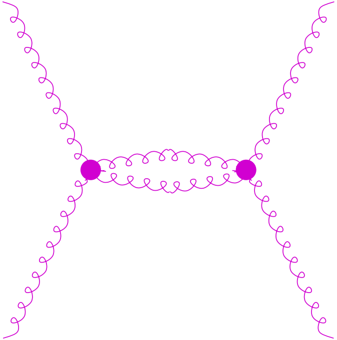
Explicitly

Cut diagram (=inelastic)



Contribution *G*

Uncut (=elastic)



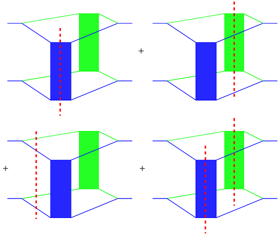
Contribution -*G*

Negative contribution means shadowing / screening

Consider two Pomerons

Total cross section contribution (at least one +G) **proportional to**

$$0 \times (-G)^2 + 2G(-G) + G^2$$



Di-jet cross section σ_{dijet}: Each cut Pomeron produces one di-jet =>

$$\sigma_{\mathrm{dijet}} = 0 \times (-G)^2 + 1 \times 2G(-G) + 2 \times G^2$$
$$= \quad 0 \quad - \quad 2G^2 \quad + \quad 2G^2 \quad = \quad 0$$

The different contributions cancel!!

Consider three Pomerons

Total cross section contribution (at least one +G) **proportional to**

$$0 \times (-G)^3 + 3G(-G)^2 + 3G^2(-G) + G^3$$

For di-jet cross section σ_{dijet}, add coefficients (number of di-jets):

$$\begin{aligned} & 0 \times (-G)^3 + 1 \times 3G(-G)^2 + 2 \times 3G^2(-G) + 3 \times G^3 \\ = \quad & 0 + 3G^3 - 6G^3 + 3G^3 = \quad 0 \end{aligned}$$

Again the different contributions cancel!!

Contribution for n Pomerons (*k* refers to the cut Pomerons):

$$\sigma_{\mathrm{dijet}}^{(n)} \propto \sum_{k=0}^{n} k\, G^{k}\, (-G)^{n-k} \binom{n}{k}$$

$$\propto \sum_{k=0}^{n} (-1)^{n-k}\, k \times \binom{n}{k}$$

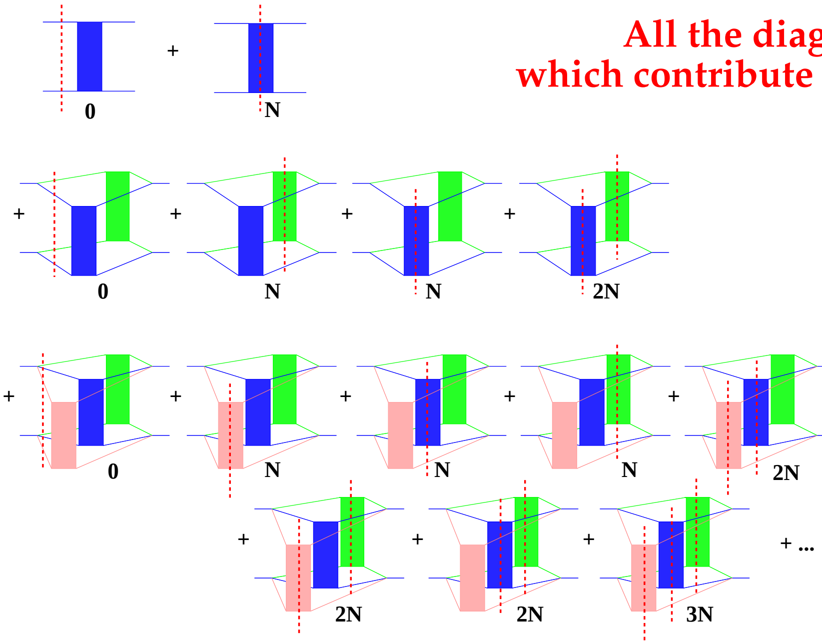
$$= 0 \quad \text{for any } n > 1$$

- ☐ **Almost all of the diagrams (i.e. n=2, n=3, ...) do not contribute at all to the inclusive cross section**

- ☐ **Enormous amount of cancellations (interference), only n=1 contributes**

- ☐ **AGK cancellations**
  (Abramovskii, Gribov and Kancheli cancellation (1973))

All the diagrams which contribute to pp

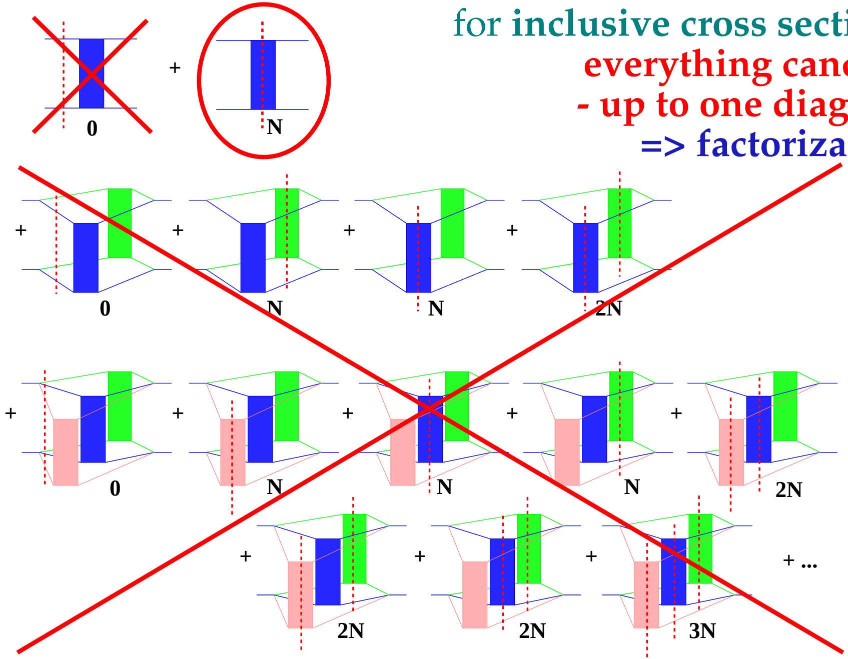


for **inclusive cross sections**

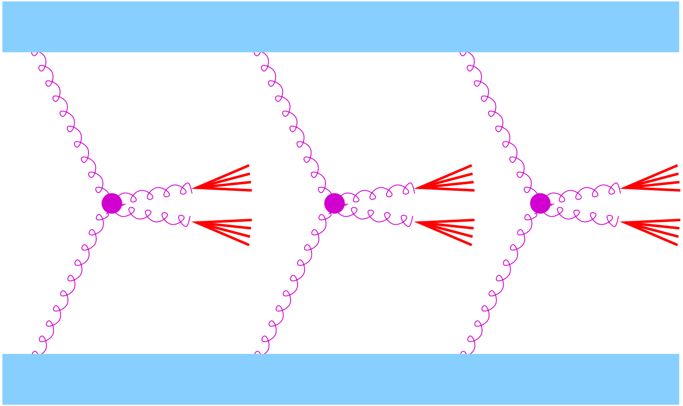
everything cancels

- up to one diagram

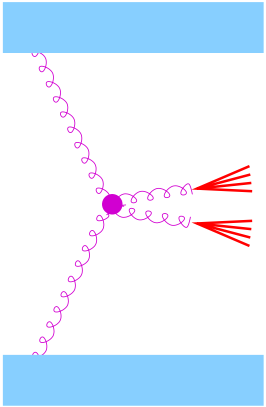
=> factorization



Even though the real events show multiple Pomerons



For inclusive cross sections (and only then) a simple diagram is enough



which corresponds to factorization:

$$\sigma_{\mathrm{incl}} = f \otimes \sigma_{\mathrm{elem}} \otimes f$$

Remark:

- ☐ **We get perfect AGK cancellations in our simplified GR picture (no energy sharing)**

- ☐ **In the full scheme, it works at large pt (in EPOS4)**

Beyond factorization

Factorization simplifies things enormously!

Extremely useful when computing inclusive di-jet cross sections to study the underlying elementary QCD processes. The full event structure is not needed.

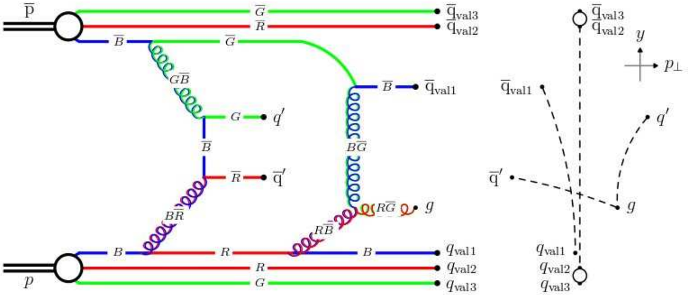
However, many observables require "full events", like everything related to given multiplicity selections.

Two strategies to deal with.

Strategy 1

Start out from factorization, sampling several di-jets from a single diagram,

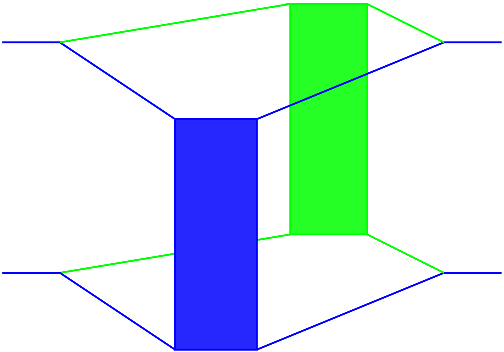
and then attribute them to different subprocesses, redefine color structures (Pythia, Herwig,...)



Strategy 2

Start out from multi-Pomeron S-matrix, sample multi-Pomeron configurations using cutting rule techniques, employing Markov chains

and sample di-jets for each Pomeron, one per Pomeron (EPOS)

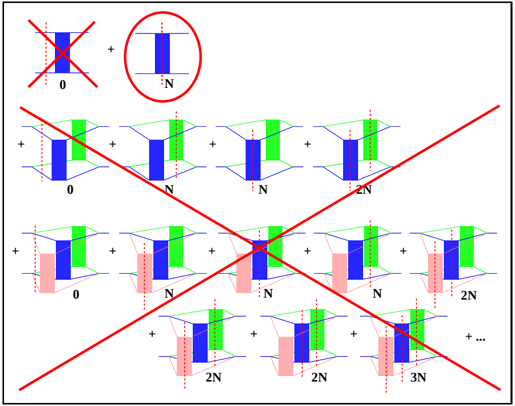


Pros and cons

Strategy	Pros	Cons
Method 1 (PYTHIA)	Simple to realise ———————— Best method for inclusive cross sections	"Reconstruction" of multiple scattering without solid theoretical basis ———————— probably not working for small pt ———————— No obvious extension towards AA
Method 2 (EPOS)	Solid theoretical basis concerning multiple parallel scattering ———————— Straightforward generalization for AA	Realisation technically demanding ———————— Factorization not for free, big effort needed to realize the cancellations

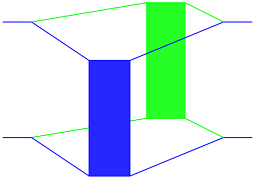
Main problem for the EPOS method:

Since all diagrams are considered:



In case of inclusice cross sections, the corresponding diagrams must actually cancel, which requires high precision and good strategies

AA collisions

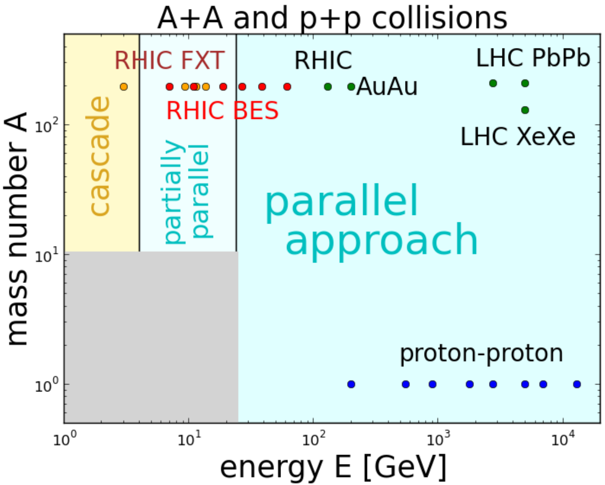


Almost trivial to extend the multiple Pomeron picture to AA.
The T-matrix is essentially a product of the pp expressions:

$$-i \prod_{\text{pairs}} \{ iT_{\text{Pom}} \times \ldots \times iT_{\text{Pom}} \}$$

Again, the difficulty is the fact that realizing AGK cancellations requires big efforts

Crucial! Amounts to binary scaling



So again, the multiple Pomeron approach is difficult (high precision and sophisicated strategies needed to get cancellations)

but there is no real alternative, we need a "parallel approach"